# Lesson 1. Introduction to Discrete Dynamical Systems

## 1 Course overview

- **Economics** is the study of how society manages its scarce resources
- In particular, economists study
  - how people make decisions – e.g. how much they work, what they buy, how much they save
  - how people interact with each other – e.g. how buyers and sellers determine the price of a good
  - how forces and trends affect the wealth and resources of society as a whole – e.g. unemployment rate, growth in average income
- **Mathematics** allows us to study problems in economics with rigor, generality, and simplicity
- This course will cover various mathematical topics essential to the study of economics

## 2 Today

- What is an economic model?
- A simple economic model: interest rates
- Discrete dynamical systems: definitions, examples

## 3 What is an economic model?

- An **economic model** is a set of **variables** and a set of **relationships** (e.g. equations) between them representing some economic process
- Models are typically abstractions of the real world
- Even a rough representation of the economic process we want to study can give us good insights
  - "All models are wrong, but some are useful." –George Box, statistician

## 4 A model for interest rates

- Let $A_n$ = amount of money we have in a savings account at year $n = 0, 1, 2, \ldots$
  - Our initial deposit is $A_0$
- Let $r$ = annual interest rate
- After 1 year, how much do we have in our savings account?

- In general, what is the relationship between $A_{n+1}$ and $A_n$?

**Example 1.** Suppose our initial deposit is $A_0 = 100$, and the interest rate is $r = 0.05$. How much do we have in our savings account after 3 years?

- In general, how much will we have in our account after $n$ years?
- One way to figure this out is to write out the relationship for increasing values of $n$ until we see a pattern:

## 5 Discrete dynamical systems

- More generally, we want to study how a quantity changes over time
- Let $A_n$ be the quantity at time $n = 0, 1, 2, \dots$
- A **discrete dynamical system** is an equation that describes a relationship between the quantity at a point in time and the quantity at earlier points in time
  - In this class, we will sometimes call these just "**dynamical systems**" or "**DS**"
- Many economic models can be represented using a dynamical system
  - e.g. the interest rate model above!

## 5.1 First-order dynamical systems

- In a **first-order** dynamical system, the quantity depends only on the quantity at the previous point in time
- Mathematically:

**Example 2.** Consider the interest rate DS

$$A_{n+1} = (1+r)A_n \qquad n = 0, 1, 2, \ldots$$

Is this a first-order DS? Why or why not?

**Example 3.** Consider the Fibonacci sequence, given by the DS

$$A_0 = 1$$
$$A_1 = 1$$
$$A_{n+2} = A_{n+1} + A_n \qquad n = 0, 1, 2, \ldots$$

What are the values of $A_2, A_3, A_4, A_5$? Is this a first-order DS? Why or why not?

- For now and the next few lessons, we will focus on first-order dynamical systems

## 5.2 Linear vs non-linear dynamical systems

- Consider a first-order DS: $A_{n+1} = f(A_n), \quad n = 0, 1, 2, \ldots$
- If $f$ is a function of the form $f(x) = sx + b$, then the DS is **linear**
- Otherwise, the DS is **nonlinear**

**Example 4.** Consider the two dynamical systems below:

$$A_{n+1} = 3A_n \qquad n = 0, 1, 2, \ldots \qquad\qquad A_{n+1} = 3A_n - A_n^2 \qquad n = 0, 1, 2, \ldots$$

Are these dynamical systems linear or nonlinear? Why?

### 5.3 Solutions to dynamical systems

- What does it mean to "solve" a DS?
- A **solution** to the DS $A_{n+1} = f(A_n)$, $n = 0, 1, 2, \ldots$ is a sequence of numbers $A_0, A_1, A_2, \ldots$ that satisfies the DS

**Example 5.** Find a solution to the DS $A_{n+1} = 3A_n$, $n = 0, 1, 2, \ldots$.

- A DS may have an infinite number of solutions!
- The **general solution** to a DS is the family of all solutions to the DS
  - For example, the general solution to the DS in Example 5 is

- An **initial condition (IC)** for a DS is a specific value of $A_0$
- A **particular solution** to a DS is a solution that also satisfies an IC

**Example 6.** Consider the DS given in Example 5, and suppose we are given the IC $A_0 = 5$. Find the particular solution that satisfies the IC.

### 5.4 Fixed points

- Suppose the DS $A_{n+1} = f(A_n)$, $n = 0, 1, 2, \ldots$ has the solution

$$A_n = c \quad n = 0, 1, 2, \ldots \qquad \text{for some real number } c \tag{$\star$}$$

- In other words:
- A DS with this solution is in an **equilibrium state**
  - The number $c$ is a **fixed point** or **equilibrium value** of the DS
  - The solution $(\star)$ is called a **constant solution**
- Finding fixed points helps us find the "natural resting points" of systems
  - e.g. market prices, national income
- How do we find fixed points of a DS $A_{n+1} = f(A_n)$, $n = 0, 1, 2, \ldots$?
  - We want:
  - We also know:
  - Therefore, we need to solve:

**Example 7.** Find the fixed points of the DS $A_{n+1} = 4A_n - 2$, $n = 0, 1, 2, \ldots$. Verify that you have found a fixed point by computing the first few values of $A_0, A_1, A_2, \ldots$.